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THE H, P AND H-P VERSION OF THE FINITE ELEMENT METHOD
BASIC THEORY AND APPLICATIONS

by

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THE H, P AND H-P VERSION OF THE FINITE ELEMENT METHOD BASIC THEORY AND APPLICATIONS

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ABSTRACT. This paper presents the survey of the basic ideas and results of the h, p and h-p versions of the finite element method.

Key words: Finite element method, the p-version of FEM, the h-p version of FEM, adaptive finite element method

1. INTRODUCTION

Finite element method is today the major tool in computational mechanics. Many research and commercial programs are used in practice. More than 40 thousand papers of theoretical and computational characters have been published.

The software of the finite element method is complex and is influenced by the theory, implementational aspects, computational experience and engineering needs. It is influenced by the hardware development and the relation between the computer and manpower cost as well as the class of problems to be solved by the engineering community. The finite element is a tool for obtaining the data of interest with the accuracy in a prescribed range in an effective way (taking into account the computer and manpower cost.) The existence of a reliable and accurate quantitative a-posteriori error estimation for any data of interest should be one of the major aspects of assessing the finite element method in general and a code in particular. The various adaptive approaches together with users interaction are essential for the effectiveness of the approach.

The majority of the results and finite element programs in structural mechanics are based on the classical h-version. Nevertheless, during the last ten years the p-version and the h-p version was developed, major theoretical results achieved and few large scale program developed. Especially, we mention here the program STRIPE (Aeronautical Research Institute of Sweden), Applied structure (Rasna Corp., CA, USA), PHLEX (Computational Mechanics, TX, USA) and MSC/PROBE MacNeal Schwendler, CA, USA). (We note that less than one hundred papers addressing the p and h-p versions have been published.)

Comparison of the methods and codes based on the h, p and h-p versions is a complex task and depends on the criteria used. It seems that one of the major criterion is the existence of a reliable quantitative error estimation of any data of interest—not only for example measured by the energy norm. In addition, the adaptive approaches are imperative, together with the robustness of the method and its effectiveness in the sense of computer and manpower cost. Theory of the method has to be well developed, so that it reliably serves to the understanding of the method and its features.

In this paper we will focus on very basic aspects of the h, p and h-p versions of the finite element method, especially with respect to the theoretical understanding. We will show that it is possible to obtain very useful understanding from the detailed mathematical analyses of a simple one dimensional problem. The mathematical theorems properly used are very important guidelines for the software in 2 and 3 dimensions too. We will present also basic results in 2 dimensions and highlight the similarity and differences between the one and two dimensional cases. In this paper we will focus on aspects and examples of most simple nature, but still keeping certain essentials of the general cases intact. We will emphasize the aspects of the p and h-p versions of the finite element method. For the survey of the practical aspect of the p-version in the context of solving complex 3 dimensional program based on program STRIPE, we refer to [2] [3]. For the survey of the basic theoretical results, we refer to [20].

In Section 2 we will address the h, p and h-p versions in one dimension, present basic very detailed theoretical results and make comments of their meaning and importance in more general setting of the adaptivity, etc., with analogs in 2 and 3 dimensions. Section 3 will elaborate on the two dimensional problems.

In this paper we will restrict ourselves, for simplicity, only to the energy norm accuracy measure, although it has relatively very limited practical engineering importance in the sense we defined above, but I will still allow us to show many basic features of the general case (discusses in

[2] [3]). Numerical examples will illustrate the use of the presented theoretical result in practical context.

2. THE FINITE ELEMENT METHOD IN ONE DIMENSION

2.1. Formulation of the problem

As said in the introduction, the one dimensional problem is a good way to explain the essentials of the h, p and h-p versions.

We will consider the following simple one dimensional model problem

$$-u'' = f \quad \text{in} \quad I = (0,1) \tag{2.1.1a}$$

$$u(0) = u(1) = 0 \tag{2.1.1b}$$

with the exact solution

$$u_0(x) = (x^{\alpha} - x), \qquad \alpha > 1/2. \tag{2.1.2}$$

Let $H^1(I)$ be the standard Sobolev space and

$$H_0^1(I) = \{u \in H_0^1(I) | u(0) = u(1) = 0\}.$$

Further, let

$$B(u,v) = \int_0^1 u'v' \, dx \tag{2.1.3}$$

be the bilinear form on $H_0^1(I) \times H_0^1(I)$ and let $\|u\|_E = (B(u,u))^{1/2}$. Obviously, $\|\cdot\|_E$ is equivalent on $H_0^1(I)$ with the Sobolev norm $\|\cdot\|_{H^1(I)}$. We will assume that $\alpha > 1/2$ to get $\|u_0\|_E < \infty$.

The solution (2.1.2) is a very good model for the two dimensional problem where the domain has corners (e.g., cracked domain, etc.) or interfaces and of 3 dimensional problems too.

In two dimensions $u = r^{\beta}\varphi(\theta) \in H^1(\Omega)$ for $\beta > 0$. This and other reasons show that the results in two dimensions with the singularity of the type $r^{\beta}\varphi(\theta)$ should be compared with the one dimensional case when $\alpha = \beta + 1/2$.

Let

$$\Delta := \{0 = x_0^\Delta < x_1^\Delta < \cdots < x_{m(\Delta)}^\Delta = 1\}$$

be the mesh on I and $I_i^\Delta = (x_{i-1}^\Delta, x_i^\Delta)$, $i = 1,\dots,m(\Delta)$, $h_i^\Delta = |I_i^\Delta| = x_i^\Delta - x_{i-1}^\Delta$, $h(\Delta) = \max h_i^\Delta$, $i = 1,2,\dots,m(\Delta)$. The points x_i^Δ will be called the nodal points and I_i^Δ the elements. Let further $p = \left(p_1^\Delta,\dots,p_{m(\Delta)}^\Delta\right)$, $p_i \geq 1$, integer be the element degree vector. We will use the notation $\Sigma = (\Delta, p)$ which will characterize the basic finite element method. Let

$$S = S(\Sigma) = \{u \in H_0(I) \,|\, u|_{I_i} \text{ is a polynomial of degree } p_i\}$$

We will call $N(\Sigma) = \dim S(\Sigma)$ the number of degrees of freedom. Finite element solution $u_\Sigma \in S(\Sigma)$ is then defined so that

$$B(u_\Sigma, v) = B(u_0, v), \quad \forall\, v \in S(\Sigma). \tag{2.1.4}$$

We will denote the error by $e = e_\Sigma = u_\Sigma - u$ and will be interested in the accuracy measure $\|\cdot\|_E$

Let us now define the h, p and h-p versions of the finite element method. To this end, assume that a sequence Σ_i with $N(\Sigma_i) = N_i$, $N_i \to \infty$ be given (resp. constructed by an adaptive approach). Then computation of u_{Σ_i} will be called

a) The <u>h-version</u>: if $\Sigma_j = (\Delta_j, p_0)$, where $p_0 = \left(p_0^{\Delta_j},\dots,p_0^{\Delta_j}\right)$, i.e., the degree of the elements are fixed and the mesh is changed (refined). This is the classical version of the finite element method.

b) The <u>p-version</u>: if $\Sigma_j = (\Delta, p_j)$, where $p_j^\Delta = \left(p_{j,1}^\Delta,\dots,p_{j,m(\Delta)}^\Delta\right)$

either $p^{\Delta}_{j,i} = p^{\Delta}_{j}$ (i.e., uniform p) or $p^{\Delta}_{j,i} \neq p^{\Delta}_{j,\ell}$ if $\ell \neq i$ (i.e., nonuniform degrees), i.e., mesh is fixed and degrees are increased.

d) The h-p version: if $\Sigma_j = (\Delta_j, p_j)$ where both, the meshes and the degrees, are changed.

We will associate to the solution u_Σ a computable a-posteriori error estimator $\|e_\Sigma\|_E$. Usually the error estimator is computed by elemental error indicator $\eta(I^{\Delta}_{i})$ with

$$\left(\sum_{i=1}^{m(\Delta)} \eta^2(I^{\Delta}_{i})\right)^{1/2} = \mathcal{E}(u_\Sigma).$$

We will assume that the goal of computation is to design Σ so that $\mathcal{E}(u_\Sigma) \approx \|e_\Sigma\|_E \leq \tau\|u\|_E$ where τ is an a-priori given tolerance (say 1% or 5%). We will not address here in detail the concrete form of the estimation.

2.2. The h-version of the finite element

Obviously we have for the h-version $N(\Sigma) = mp - 1$. The first theoretical question is what is the best accuracy the h-version could provide among all meshes. We have

Theorem 2.2.1 [36]. Let $\alpha > 1/2$ be a non-integer, and u_0 be given by (2.1.2), then there is a constant $C = C(\alpha,p) > 0$ such that for any mesh

$$\|e_\Sigma\|_E \geq CN^{-p} \tag{2.2.1}$$ □

This theorem shows that (asymptotically) it is not possible to expect anything better than the rate N^{-p} when elements of degree p is used. This does not mean that this accuracy will be achieved for any mesh.

The question is whether for a proper sequence of meshes we get

$$\|e_\Sigma\|_E \leq CN^{-p}.$$

Hence, the problem arises, what is the best theoretical mesh. To this end we introduce the notion of the <u>grading function</u>. Function $g(x)$, $0 \leq x \leq 1$ is called a mesh grading function if the nodal points of the mesh are such that

$$x_i^\Delta = g\left(\frac{i}{m}\right), \quad i = 1,2,\ldots,m = m(\Delta)$$

The mesh will be called <u>radical</u> if $g(x) = x^\beta$. We shall assume that the grading function $g(x)$ satisfies the following conditions:

G(1): $g(0) = 0$, $g(1) = 1$,

G(2): g is continuous and strictly increasing,

G(3): $g \in C^1(0,1) \cap C^0[0,1]$.

Then we have

<u>Theorem 2.2.2</u> [36]. Among all grading functions $g(t)$ satisfying Assumption G(1) – G(3)

$$g_{opt} = x^\beta, \quad \beta = \frac{p+1/2}{\alpha-1/2}$$

is the optimal one. Precisely with this grading function

$$\text{(2.2.2)} \qquad \lim_{m\to\infty} m^p \|e_\Sigma\|_E = C(\alpha,p)\left(\frac{p+\frac{1}{2}}{\alpha-\frac{1}{2}}\right)^{p+\frac{1}{2}}$$

where

$$\text{(2.2.3)} \qquad C(\alpha,b) = \frac{\alpha\Gamma(\alpha)|\sin \pi\alpha|}{\sqrt{\pi}} \frac{\Gamma(p-\alpha+1)}{4^p\sqrt{2p+1}\ \Gamma(p+\frac{1}{2})}$$

attains the minimum. We see that the optimal mesh is radical one and gives maximal possible rate N^{-p}. □

<u>Theorem 2.2.3</u> [36]:

a) If $\beta > \frac{p}{\alpha-\frac{1}{2}}$, then

(2.2.4) $$\lim_{m\to\infty} m^p \|e_\Sigma\|_E = \frac{C(\alpha,p)\beta^{p+\frac{1}{2}}}{\sqrt{(2\alpha-1)\beta-2p}}$$

b) If $\beta_{cr} = \frac{p}{p-\frac{1}{2}}$, then

(2.2.5) $$\lim_{m\to\infty} \frac{m^p \|e_\Sigma\|_E}{\sqrt{\ell n\ m}} = C(\alpha,p)\beta^{p+\frac{1}{2}}$$

c) If $\beta < \frac{p}{\alpha-\frac{1}{2}}$, then

(2.2.6) $$\lim_{m\to\infty} m^p \|e_\Sigma\|_E = C_1(\alpha,\beta,p)$$

where $C(\alpha,p)$ is defined by (2.2.3) and $0 < C_1(\alpha,\beta,p) < \infty$ has a more complicated expression. □

We remark that $\beta = 1$ corresponds to the uniform mesh. Theorems 2.2.1-2.2.3 indicate

a) The maximal achievable rate of convergence is N^{-p} and this rate can be achieved by a proper mesh also for nonsmooth solution of the type (2.2.1). This optimal mesh depends on the strength of the singularity and the used degrees of the elements. Increasing the degrees of elements leads to the strengthening of the refinement.

b) The optimal mesh is the radical one. Underrefinement is more "dangerous" than overrefinement, because it can significantly slow down the convergence.

We have addressed the method and the optimal mesh only in the case when the solution is given by (2.2.1). In general, the mesh has to be constructed by an adaptive procedure. Two types of adaptive mesh construction are used (in one or 2, 3 dimensions).

a) A sequence of meshes in constructed by consecutive refinement or derefinement of the particular elements. The goal for refinement or derefinement is to make the (elemental) error indicators η approximately equal. For a detailed theoretical analysis of this approach in one dimension, we refer to [23]. The final mesh is accepted when the error estimator indicates the acceptable accuracy.

b) The initial usually crude mesh is used for the computation of the basic characteristic of the solution. Using these characteristics a new mesh is constructed with the aim to get acceptable accuracy. If the accuracy is not achieved, the process is repeated. (See. e.g. [6][39][40][57][59])

We will call the adaptive approach based on a) the <u>local approach</u> and if it is based on b) <u>the global approach</u>.

The mentioned theoretical results give an important insight into the design of the adaptive procedure and judging its performance when used on the benchmark problem with the solution (2.2.1). The notion of the grading function can be used in general and especially in the global adaptive approach. See [39] [40].

The mesh construction is very influenced by the implementational aspects. For example, in the local adaptive approach often only the halving of particular elements is used. Accuracy and reliability control is a very important task in the practice of finite element method. Various mathematical and heuristic methods for 1, 2 and 3 dimensional problems were suggested. We refer here as examples to [1][5][6][14][16][17][18][19][24][25][30][31][32][34][40][41][47][48][53][54][55][56][57][58][60][51] and the survey [17]. Various adaptive codes based on the h-version were written (see. e.g. [26] [33] [44] [51] [59]). The error estimates are in the h-version usually for the energy norm error. The a-posteriori estimates for other data as stresses in a point, the stress intensity factor, etc., are almost impossible to get by the approaches used for the energy norm measure.

2.3. The p-version

In the p-version the mesh Δ is fixed, i.e., the mesh is made usually by the user (based on his experience and various basic rules) and the accuracy is obtained by (adaptively) increasing the degrees of elements. The degrees could be uniform or nonuniform. The error estimator $\mathcal{E}$ is usually based on comparison of computed data for the sequence of used degrees and an extrapolation procedure. See, e.g. [54,55,56] for details.

In 2 and 3 dimensions, the user identifies the singular areas as corners and edges, and, for example, constructs locally refined meshes in these singular areas, and then, by a general mesh generator, connects (complements) these local meshes to the mesh of the entire domain or constructs first the mesh and then refines it in critical areas. The degree of the elements which leads to the desired accuracy is then adaptively determined. The singularity refinement is made by the "layers" of the elements around the singularity. (See examples of two dimensional meshes of this type in Section 3 (Fig. 3.2.1)). A minimum of 2-3 layers is practically necessary because for accurate stress computation in a point at least 2 layers have to separate the point from the singularity point. The p-version allows to use relatively very small number of elements.

Let us now present typical results showing the features of the p-version when applied to the solution (2.1.2). To this end let

$$\bar{\eta}(I_i^\Delta) = \inf_{\omega = P_p(I_i^\Delta)} \|u_0 - \omega\|_{E(I_i^\Delta)} = \|u - u_\Sigma\|_{(I_i^\Delta)} \tag{2.3.1}$$

where $P_p(I_i^\Delta)$ is the class of all polynomials of degree p. We mention that (2.3.1) is in our case the error of the finite element solution and $\|e_\Sigma\|_E^2 = \sum_{i=1}^{m} \bar{\eta}(I_i^\Delta)$.

<u>Theorem</u> <u>2.3.1</u> [36]. Let $I_i^\Delta = (a,b)$ then

a) if $0 < a < b \le 1$

$$(2.3.2) \qquad \bar{\nu}(I_i) = C_1(\alpha)(b-a)^{\alpha-\frac{1}{2}} \left(\frac{1-r_i^2}{2r_i}\right)^{\alpha-1} \frac{r_i^p}{p^\alpha}\left(1+0\left(\frac{1}{p^\sigma}\right)\right)$$

where $\sigma > 0$ and

$$(2.3.3) \qquad C_1(\alpha) = \frac{1}{2^{\alpha-(1/2)}} \frac{\alpha\Gamma(\alpha)|\sin \pi\alpha|}{\sqrt{\pi}}$$

$$(2.3.4) \qquad r_i = \frac{b^{1/2}-a^{1/2}}{b^{1/2}+a^{1/2}}$$

c) If $a = 0$, $0 < b \le 1$

$$\eta(I) = C_0(\alpha)b^{\alpha-\frac{1}{2}} \frac{1}{p^{2\alpha-1}} 1 + 0\left(\frac{1}{p}\right)$$

where

$$(2.3.3a) \qquad C_0(\alpha) = \frac{\alpha\Gamma(\alpha)^2|\sin \pi\alpha|}{\pi\sqrt{2\alpha-1}} .$$ □

Theorem 3.3.1 shows that in the first element I_1 the rate is only algebraic, while in all others it is exponential. It shows also that the accuracy in the first element I_1^Δ is essentially governed by its size $(h_1 = b)$ while the accuracy of the other elements is governed by their degree (because the exponential rate). We see that the error $\bar{\eta}(I_i^\Delta)$, $i \ge 2$ will be essentially equal (for uniform p) in all the elements I_i^Δ when the mesh is geometric with the factor q, i.e., $x_{i-1} = qx_i^\Delta$, $x_{m(\Delta)}^\Delta = 1$. Hence, a geometric mesh is a good choice (see Section 2.2.4 for additional arguments).

The question how many elements and which factor q should be selected is very important for practical computation. Small q and large $m(\Delta)$ can lead to the "overrefined" mesh for p small because the error $\|e\|_E$ is governed by the error in the largest element. For high p the small factor q is preferable because the error is essentially governed by the elements close to the singularity. The overrefinement is advisable because when small p (which is adaptively determined) leads to the desired accuracy, the computation is cheap and overrefinement does not essentially increase the cost. On the other hand, if the desired accuracy needs high p, the underrefinement could be very costly. Hence, the user designing a mesh has to take a "cost risk" although he will get always the required accuracy by adaptive solution of the degrees.

To illustrate these points, consider the mesh with 2 elements $I_1 = (0,\rho)$, $I_2 = (\rho,1)$ and compute (see (2.3.2))

$$\eta_*(I_1) = \rho\delta^{\alpha-\frac{1}{2}} \frac{1}{p^{2\alpha-1}} \tag{2.3.5}$$

$$\eta_*(I_2) = (1-\rho)^{\alpha-\frac{1}{2}} \frac{r_2^p}{p^2} \tag{2.3.6}$$

for various p, α and δ. Note that in (2.3.5) and (2.3.6) we did not include the constants $C_0(\alpha)$ and $C_1(\alpha)$ which have the factor $|\sin \pi\alpha|$. To model the case, which we will study in Section 3 $(u \approx r^{1/2})$ we will use $\alpha = 1,2,3$. This has to be understood as the limiting case $\alpha = 1-\varepsilon, 2-\varepsilon, 3-\varepsilon$ with $\varepsilon \to 0$ or when the approximate solution is $u^* = x^\alpha \lg x$.

In Table 2.3.1, we report $\eta_*^2(I_1)$ and $\eta_*^2(I_2)$ as function of ρ, p and $\alpha = 1,2,3$. We see, for example, that for $\alpha = 1$ the error is governed by the error in I_1 for $p \geq 2$, and hence ρ is "too large", i.e., the mesh is underrefined. For $\alpha = 2$, the mesh is "overrefined" when $\rho = 0.1$ and $p \leq 8$, and for 0.2 when $p \leq 5$. Noting the magnitude of the errors, we see that the overrefinement makes no harm because the accuracy is achieved for small p, and computation is cheap. From (2.3.5) and (2.3.6) as well as from Table 2.3.1, we see that(for the fixed mesh) for smaller p the rate is exponential and for p larger when I_1 is error governing interval, the rate

TABLE 3.1. The errors $\eta_*(I_2^2)$, $\eta_*(I_2^2)$.

$\alpha = 1$						
	$\rho = 0.1$		$\rho = .15$		$\rho = 0.5$	
p	$\eta_*^2(I_1)$	$\eta_*^2(I_2)$	$\eta_*^2(I_1)$	$\eta_*^2(I_2)$	$\eta_*^2(I_1)$	$\eta_*^2(I_2)$
1	1.00-1	2.42-1	1.50-1	1.65-1	5.01-1	1.47-2
2	2.50-2	1.63-2	3.75-2	8.08-3	1.25-1	1.08-4
3	1.11-2	1.96-3	1.66-2	7.00-4	5.55-2	1.42-6
4	6.25-3	2.98-4	9.37-3	7.69-5	3.12-2	2.34-8
5	4.00-3	5.15-5	6.00-3	9.59-6	2.00-2	4.42-10
6	2.77-3	9.65-6	4.17-3	1.30-6	1.38-2	9.04-12
7	2.04-3	1.91-6	3.06-3	1.86-7	1.02-2	1.95-13
8	1.56-3	3.95-7	2.34-3	2.78-8	7.81-3	4.41-15

$\alpha = 2$						
	$\rho = 0.1$		$\rho = 0.2$		$\rho = 0.5$	
p	$\eta_*^2(I_1)$	$\eta_*^2(I_2)$	$\eta_*^2(I_1)$	$\eta_*^2(I_2)$	$\eta_*^2(I_1)$	$\eta_*^2(I_2)$
1	1.00-3	1.96-1	3.83-3	1.19-1	1.25-1	3.67-3
2	1.56-5	3.31-3	5.27-5	1.46-3	1.95-3	6.77-6
3	1.37-6	1.77-4	4.63-6	5.62-5	1.71-4	2.93-8
4	2.44-7	1.51-5	8.24-7	3.47-6	3.05-5	3.66-10
5	6.40-8	1.66-6	2.16-7	2.77-7	8.00-6	4.42-12
6	2.14-8	2.17-7	7.23-8	7.23-8	2.68-6	6.27-14
7	8.50-9	3.65-8	2.81-8	2.74-9	1.06-6	9.97-16
8	3.81-9	5.01-9	1.29-8	3.14-10	4.76-7	1.72-17

TABLE 3.1. The errors $\eta_*(I_2^2)$, $\eta_*^2(I_2)$.
(Continued)

$\alpha = 3$						
	$\rho = 0.1$		$\rho = .15$		$\rho = 0.5$	
p	$\eta_*^2(I_1)$	$\eta_*^2(I_2)$	$\eta_*^2(I_1)$	$\eta_*^2(I_2)$	$\eta_*^2(I_1)$	$\eta_*^2(I_2)$
1	1.00-5	1.59-1	7.59-5	8.65-2	3.12-2	9.19-4
2	9.76-9	6.71-4	7.42-8	2.63-4	3.05-3	4.23-7
3	1.69-10	1.59-5	1.28-9	4.51-6	5.29-7	1.09-9
4	9.53-12	7.64-7	7.24-11	1.56-7	2.98-8	5.72-12
5	1.02-13	5.41-8	7.77-12	8.01-9	3.20-9	4.24-14
6	1.65-13	4.88-9	1.26-12	5.23-10	5.16-10	4.35-16
7	3.59-14	5.23-10	2.69-13	4.05-11	1.10-10	5.08-18
8	9.33-15	6.33-11	7.07-14	3.54-12	2.91-11	6.72-20

is algebraic. The transition point between the exponential and algebraic rate is when the errors in I_1 and I_2 are equal. We will see the same features in 2 dimensions.

Theorem 2.3.1 and Table 2.3.1 show also that nonuniform degree distribution is optimal. If the mesh is underrefined, then high p in I_1 and low in I_2 is optimal and for overrefined mesh the opposite holds. The adaptive code for nonuniform degrees constructs such distribution. Once more we see that overrefinement is preferable. The value of $\rho \approx 0.15$ is, as we will see in the next section, optimal for all α when nonuniform p is used and practically is good also for p uniform. The same effects hold in 2 dimensions as will be seen in Section 3.

As was said above, the mesh design is very important. We refer to [2] [3] [54] [55] [56] for practical rules for a design as a "good" mesh. Let us underline—as we have seen—that meshes for the p-version have different character than that of the h-version with fixed degree p. Nevertheless, sometimes an adaptive construction of the mesh for the h-version with, say $p = 3$, could be possibly used for the p-version although it is far from the optimal [59]. This could be seen from comparison of the formulae for the optimal mesh for the h-version and $p = 3$ discussed in Section 2.2 and the formulae of Section 2.3. The error estimator based on the extrapolation is very accurate and robust. We will see it more in 2 dimensional example.

The first theoretical results for the p-version in 2 dimensions are in [22], for the three dimensional problems in [28] [29]. For the survey of the results, we refer to [20]. The p-version is related to the spectral element method. See, e.g. [43].

2.4. The h-p version

In this section we will discuss theoretical questions about the h-version. The theoretical results will also give an additional insight into the desirable mesh design for the p-version.

The first question is about the lower bound of the error.

Theorem 2.4.1 [36]. For any mesh Δ with $m(\Delta)$ elements and any $p = \left(p_1^\Delta, \dots, p_{m(\Delta)}^\Delta\right)$ degree vector we have

$$\|e_\Sigma\| \geq C(\alpha)\,\frac{q_0^{\sqrt{\mu N}}}{\sqrt{N^\mu}} \tag{2.4.1}$$

where

$$\mu = \alpha - 1/2, \quad q_0 = (\sqrt{2} - 1)^2 \approx 0.17. \tag{2.4.2}$$

(Here once more the solution i_u is given in (2.2.1)). □

The next question is whether the error (2.4.1) can be achieved.

Theorem 2.4.2 [36]: Let the mesh Δ be geometric with the factor q, i.e. $x_0 = 0$, $x_i^\Delta = q^{m(\Delta)-i}$, $i = 1,2,\ldots,m(\Delta)$ and let $p_i = [1+s(i-1)]$ (where $[\cdot]$ means the integral part). Then we have

1. if $s > s_0$, then

(2.4.3) $$\|e_\Sigma\| \approx C(\alpha,q,s)q^{(\alpha-\frac{1}{2})\sqrt{2N/s}}$$

2. if $s < s_0$, then

(2.4.4) $$\|e_\Sigma\|_E \approx C(\alpha,q,s)r^{\sqrt{2Ns}}$$

3. if $s = s_0$, then

(2.4.5) $$\|e_\Sigma\| \approx C(\alpha,q)e^{-\sqrt{(\alpha-\frac{1}{2})N}\,\sqrt{2\,\ell n\,q\,\ell n\,r}}$$

and

(2.4.6) $$r = \frac{1-\sqrt{q}}{1+\sqrt{q}}, \quad s_0 = (\alpha - 1/2)\,\frac{\ell n\,q}{\ell n\,r}\,.$$

Furthermore, the optimal geometric mesh and linear degree vector combination is given by

(2.4.7) $$q_{opt} = (\sqrt{2} - 1)^2$$

$$s_{opt} = 2\alpha - 1$$

In this case

(2.4.8) $$\|e_\Sigma\|_E \approx C(\alpha)[(\sqrt{2} - 1)^2]^{\sqrt{(\alpha/\frac{1}{2})N}}$$

In (2.4.3) and (2.4.7) $\sim$ means equivalency with the constants depending on the (α,q,s), (α,q) and α, respectively. □

Comparing (2.4.1) and (2.4.3) we see that up to the algebraic term $\sqrt{N^\mu}$ in (2.4.1) the error for the geometric mesh is optimal (i.e., the geometric mesh can be in this sense understood to be optimal).

Theorem 2.4.1 and (2.4.2) show that geometric mesh with $q \approx 0.17$ is optimal independently of the strength of the singularity α provided that degrees of the elements are not constants and their distribution depend on the strength of the singularity. For the uniform degree distribution and geometric mesh, we have

<u>Theorem</u> <u>2.4.3</u> [36] Let Δ be a geometric mesh with the factor q and m elements and uniform degrees $p = sm(\Delta)$. Then we have

1) If $s > s_0$

$$\|e_\Sigma\|_E \approx C(\alpha,q)\,\frac{q^{(\alpha-\frac{1}{2})\sqrt{N/s}}}{\sqrt{sN^{2\alpha-1}}} \tag{2.4.9}$$

2) If $s < s_0$

$$\|e_\Sigma\|_E \sim C(\alpha,q)\,\frac{r^{\sqrt{sN}}}{\sqrt{sN^{\alpha}}} \tag{2.4.10}$$

3) If $s = s_0$, then

$$\|e_\Sigma\|_E \sim C(\alpha,q)\,\frac{e^{\sqrt{(\alpha-\frac{1}{2})\,N\ell n\, r\, \ell n\, q}}}{\sqrt{N^{\sigma}}} \tag{2.4.11}$$

is optimal for given q with

$$s_0 = \frac{(\alpha-\frac{1}{2})\ell n\, q}{\ell n\, r}, \qquad r = \frac{1-\sqrt{q}}{1+\sqrt{q}}$$

$$\sigma = \min(2\alpha-1,\alpha).$$

The optimal combination is given by

$$q = q_{opt} = (\sqrt{2}-1)^2 \tag{2.4.12}$$

$$s = s_{opt} = 2\alpha - 1 \tag{2.4.13}$$

□

Theorem 2.4.3 shows that the factor q_0 of the geometric mesh is optimal for uniform mesh but the exponential rate of the convergence is not optimal. The exponent is by the factor $\sqrt{2}$ smaller. We see also that the increase of the degrees in analogous as in the case of nonuniform degrees.

Theorem 2.4.3 does not apply that for the uniform degree the geometric mesh is optimal. In fact, we have ween in Section 2.2 that for every degree p the radical mesh is optimal. Hence, we can ask the question, what rate of convergence will be obtained when the optimal racial meshes with optimal number of elements are used and $p \to \infty$?

Theorem 2.4.4 [36]. There exists sequence of meshes Δ (dependent on p) such that for uniform degrees of elements we have

$$\|e_{\Sigma}\|_E \sim \frac{1}{\sqrt{N^{\alpha-1}}} e^{-4\sqrt{(\alpha-1/2)N}\ /e} \tag{2.4.14}$$

This error is obtainable by using the proper sequence of radical meshes with $p \approx \sqrt{4(\alpha-1/2)N}/e$ as $N \to \infty$, $\beta = \frac{p+1/2}{\alpha-1/2}$. Let us summarize the theorem in Table 2.4.1. For $r = (1-\sqrt{q})/(1+\sqrt{q})$ we have

$$\|e_{\Sigma}\|_E \sim \frac{1}{\sqrt{N^{\sigma}}} e^{\kappa\sqrt{(\alpha-1/2)N}} . \tag{2.4.15}$$

TABLE 4.1. The error for the h-p version.

METHOD	q	s	κ	σ
Geometric mesh nonuniform p-distribution Theorem 2.4.2	q	$(\alpha-1/2)\frac{\ln q}{\ln r}$	$\sqrt{2 \ln q \ln r}$	0
	1/2	$0.3932(\alpha-1/2)$	1.5632	0
	$(\sqrt{2}-1)^2$	$2\alpha - 1$	1.7627	0
Geometric mesh uniform p-distribution Theorem 2.4.3	q	$(\alpha-1/2)\frac{\ln q}{\ln r}$	$\sqrt{\ln q \ln r}$	$\min(\alpha, 2\alpha-1)$
	1/2	$0.3932(\alpha-1/2)$	1.1054	$\min(\alpha, 2\alpha-1)$
	$(\sqrt{2}-1)^2$	$2\alpha - 1$	1.2464	$\min(\alpha, 2\alpha-1)$
Radical mesh q and s asymptotic Theorem 2.4.4			1.4715	$\alpha - 1/2$

Let us now briefly analyze Table 4.1. The h-p version leads to the exponential rate of convergence $e^{-\kappa\sqrt{(\alpha-1/2)N}}$. For the uniform p and optimal geometrical mesh $\kappa = 1.24$ while for the optimal nonuniform mesh we get $\kappa = 1.76 = 1.24\sqrt{2}$. Analogous results hold in 2 dimensions when the uniform degrees leads to the exponents which is by factor $\sqrt[3]{3}$ smaller than the exponent for the nonuniform degree distribution on a geometric mesh. For uniform degrees, a sequence of the meshes with the exponent 1.47 can be constructed. The shown results indicate roughly what kind of meshes should be considered for the mesh design for the p-version .

The mentioned results have to be interpreted in the light of adaptive approaches when applied for practical computations.

Once more two strategies of the mesh generation could be used. The local and global one as discussed in Section 2.2. In the local approach the main difficulty is to decide whether change of the degree or refinement (or derefinement) of an element has to be made. See, e.g. [36] [48] [50] for a discussion. In the global strategy, the main difficulty is to extract from the solution the information allowing to design global mesh and global degree distribution. The adaptive approach based on <u>p uniform</u> which is made as a sequence of meshes for increasing uniform degrees (analogously as in Theorem 2.4.4) seems too impractical.

The h-p version was theoretically analyzed in the two dimensional cases in [4][6][8][9][16][11][15][21][36][37][38][48][55][59]. See also the survey [20].

3. THE FINITE ELEMENT METHOD IN TWO DIMENSIONS

3.1. Introduction

In this section we will discuss the performance of the p and h-p version for the elliptic boundary value problems with piecewise analytic data. We will consider polygonal domains with straight or curved sides, the piecewise analytic coefficients of the operator, piecewise analytic right hand side and boundary conditions which type could change. This class of problems is typical in engineering

It is well known that the solution has singular behavior in the corners of the domain, places where the boundary condition abruptly changes, etc. See [27][37][42]. Although the solution behaves in this areas similarly as the function (2.1.2) in Section 2.1, the situation here is much complex. Mathematical tools of accurate description of the regularity of the solution are needed. Such tool is the weighted Sobolev space $H^{k,\ell}_{\beta}(\Omega)$ and the countably normed space $B^{\ell}_{\beta}(\Omega)$. The use of these spaces allows to prove very accurate estimates of the error of the p and h-p version of the finite element method which are very analogous to the one cited in Section 2, although not so detailed. To asses the detailed applicability of the theoretical results for practical computation, we will present various numerical examples and tests.

The h-p and p versions in practical environment have to be understood in a context of adaptive procedure with an error estimator. We will address these aspects too.

We will not elaborate here on the h-version because this is a classical approach and many results are available.

3.2. The spaces $H^{k,\ell}_{\beta}(\Omega)$ and $B^{\ell}_{\beta}(\Omega)$ and the model problem

Let $\Omega \subset \mathbb{R}^2$ be a polygon with vertices A_i and (open) edges $\Gamma_\ell = A_\ell A_{\ell+1}$, $1 \le \ell \le M$, $(A_{M+1} = A_1)$ and $\partial\Omega = \bigcup_{\ell=1}^{M} \bar{\Gamma}_i$. By ω_i we denote the interior angle of A_e and assume $0 < \omega_\ell \le 2\pi$. Let $r_\ell\,(x) = \mathrm{dist}(x, A_\ell)$ and $\beta = (\beta_1, \ldots, \beta_M)$ be M-tuple of real numbers $\beta_\ell \in (0,1)$. Further, for any integer k we define $\Phi_{\beta+k}(x) = \prod_{\ell=1}^{M} r_\ell^{k+\beta_\ell}(x)$.

Let $H^k(\Omega)$ be the standard Sobolev space. Then for $k \ge \ell \ge 0$, integers we define the space $H_\beta^{k,\ell}(\Omega)$ as the completion of the set of all infinitely differentiable functions under the norm

$$\|u\|^2_{H_\beta^{k,\ell}(\Omega)} = \|u\|^2_{H^{\ell-1}(\Omega)} + \sum_{\ell\le|\alpha|\le k} \|\Phi_{\beta+|\alpha|-\ell}\, D^\alpha u\|^2_{L_2(\Omega)}, \quad \ell \ge 1$$

and

$$\|u\|^2_{H_\beta^{k,0}} = \sum_{0\le|\alpha|\le k} \|\Phi_{\beta+|\alpha|-\ell}\, D^\alpha u\|^2_{L_2(\Omega)}.$$

We further introduce the countably normed space $B_\beta^\ell(\Omega)$

$$B_\beta^\ell(\Omega) = \{u | u \in H_\beta^{k,\ell}(\Omega), \ \forall\ k \ge \ell, \ \|\Phi_{\beta+|\alpha|-\ell}\, D^\alpha u\|_{L_2(\Omega)} \le Cd^{k-\ell}(k-\ell)!,$$

$$|\alpha| = k, k+1, \ldots, \ \text{constants } C \ge 1 \text{ and } d \ge 1 \text{ independent of } k\}$$

Remark 3.2.1. The intuition behind the definition of the spaces $H_\beta^{k,\ell}(\Omega)$ and $B_\beta^\ell(\Omega)$ is to consider function $u = r^\gamma$ in the neighborhood of the corners and achieve that it naturally belongs to these spaces. The solution u behaves in the neighborhood of the corners similarly as this function. See [7][11][12][37].

Let us consider the following model problem.

$$(3.2.1) \qquad \begin{aligned} -\Delta u &= f \quad \text{on } \Omega \\ u|_{\Gamma_D} &= g_D = G_D|_{\Gamma_D} \\ u|_{\Gamma_D} &= g_N = G_N|_{\Gamma_N} \end{aligned}$$

where $\Gamma_D = \bigcup_{i\in\mathcal{D}} \bar{\Gamma}_i$, $\Gamma_N = \bigcup_{i\in\mathcal{N}} \Gamma_i$; $\mathcal{D} \ne 0$ is a subset of $\mathcal{M} = \{1,2,\ldots,M\}$ and $\mathcal{N} = \mathcal{M} - \mathcal{D}$. Let $H_D^1(\Omega) = \{u | u \in H^1(\Omega), u = 0 \text{ on } \Gamma_D\}$. The weak form of (3.2.1) is $u \in H^1$, $u = G_D$ on Γ_D and such that for any $v \in H_D^1(\Omega)$

(3.2.2) $$B(u,v) = \int_\Omega fv\,dx + \int_{\Gamma_N} G_N v\,ds = F(v)$$

with

(3.2.3) $$B(u,v) = \int_\Omega \nabla u \nabla v\,dx$$

Similarly, as in one dimension, we define the energy norm

(3.24) $$\|u\|_E = (B(u,u))^{1/2}$$

which is equivalent to $\|\cdot\|_{H^1(\Omega)}$ on $H_D^1(\Omega)$.

Consider now the domain Ω shown in Fig. 3.2.1. Then we have

Theorem 3.2.1 [7]. If $f \in H^{k,0}_{\tilde{\beta}}(\Omega)$, $G_D \in H^{k+2,2}_{\tilde{\beta}}(\Omega)$, $G_N \in H^{k+1,1}_{\tilde{\beta}}(\Omega)$ with $\tilde{\beta}_i \in (0,1)$, $1 \le i \le M$, then problem (3.2.1) has unique solution $u \in H^{k+2,2}_{\beta}(\Omega)$ with $\beta_i = \tilde{\beta}_i$ if $\tilde{\beta}_i > 1 - \kappa_i$ or $\beta_i \in (1-\kappa_i, 1)$ if $\tilde{\beta}_i \le 1 - \kappa_i$, where $\kappa_i = \frac{\pi}{\omega_i}$ if $\Gamma_i, \Gamma_{i+1} \subset \Gamma_D$ (or Γ_N) and $\kappa_i = \frac{\pi}{2\omega_i}$ otherwise.

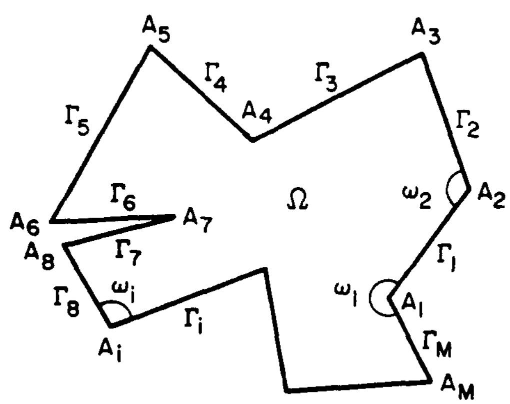

Fig. 3.2.1. Polygonal domain Ω.

Further, if $f \in B^0_{\tilde{\beta}}(\Omega)$, $G_D \in B^2_{\tilde{\beta}}(\Omega)$ and $G_N \in B^1_{\tilde{\beta}}(\Omega)$, then $u \in B^2_{\beta}(\Omega)$.

Remark 3.2.2. The weight β_i depends on $\tilde{\beta}_i$ and κ_i which/is related to the geometry of Ω. Hence, also if f, G_D and G_N are analytic, the solution still may have singular behavior because of the unsmootheness of the boundary Γ of Ω. □

Remark 3.2.3. If Ω is a polygon with the analytic curvilinear edges, the solution u of (3.2.1) will be in $B^2_{\beta+\varepsilon}(\Omega)$, $\varepsilon > 0$ (instead of $B^2_{\beta}(\Omega)$) with $\varepsilon > 0$ arbitrary (see [7])

Remark 3.2.4. Theorem 3.2.1 although formulated only for the problem (3.2.1) holds also for other problems as for the elasticity problems where the values of β_i and $\tilde{\beta}_i$ are properly adjusted (i.e., they are not the same as in Theorem 3.2.1). Analogous results are also valid for the eigenvalue problems, etc. For more, see [12].

In the next section we will address the problem (3.2.1) and analogous problems for the elasticity and will always assume that $G_D = 0$.

3.3. The h-p version

The h-p version simultaneously refines the mesh and increases the degrees of the elements. In this section we assume that the solution u of the problem has singularity <u>only</u> <u>at</u> <u>one</u> <u>vertex</u> of the domain. Let us define in the neighborhood ω of the vertex the geometrical mesh $\Omega^m_\sigma = \{\Omega_{ij}, \ 1 \le i \le m, \ 1 \le j \le J(i)\}$ where Ω_{ij} are the elements, the triangles or quadrilaterals (straight or curved). Example of a sequence of geometrical meshes is shown in Fig. 3.3.2. The elements Ω_{ij} satisfy the usual conditions preventing their degeneration and other standard conditions.

We denoted by σ the mesh factor (denoted in Section 2 by q) and by m the number of layers. Elements Ω_{ij}, $j = 1,2,\ldots,J(i)$ are located in the (i-1)-th layer. Denote by d_{ij} the distance between the element Ω_{ij} and the vertex and by h_{ij} and $\underline{h}_{ij}$ the maximal and minimal side of the element Ω_{ij}. We will assume that $d_{ij} \approx h_{ij} \approx \underline{h}_{ij} = \sigma^{m-i}$, $1 < i \le m$, $1 \le j \le J(i)$,

$d_{1j} = 0$, $h_{1j} \approx \underline{h}_{1j} \approx \sigma^{m-1}$, $1 \le j \le J(1)$. Further, we will call the vector $p = \{p_{ij}, \ 1 \le i \le m, \ 1 \le j \le J(i)\}$, $p_{ij} > 1$ integer the degree vector. We defined only the elements Ω_{ij} in the m layers near the vertex covering ω. On $\Omega - \omega$ we complement the mesh $\{\Omega^m_\sigma\}$ by the elements $\{\Omega_r\} = {}^c\Omega^m_\sigma$. Let $\{M_{ij}\}$ be the mappings of the standard elements S onto Ω_{ij} resp. onto Ω_r. We will assume that these mappings satisfy the usual conditions of the finite element method. The finite element space $S^{p,1}(\Omega)$ is then defined by

$$S^p(\Omega^m_s) = \{\varphi | \varphi|_{\Omega_{ij}} = \varphi_{ij}(x,y) = \Phi(M^{-1}_{ij}(x,y)),$$

$$\Phi(\xi,\eta) \text{ is a polynomial of degree } p_{ij} \text{ on } S\}$$

and analogously on $\{\Omega_r\}$ with the degree $p_r = \max p_{ij}$; $S^{p,1}(\Omega) = (S^p(\Omega^m_s) \oplus S^p({}^c\Omega^m_s)) \cap H^1(\Omega)$ and $\mathring{S}^{p,1}(\Omega) = S^{p,1}(\Omega) \cap H^1_D(\Omega)$. As usual, the finite element solution $u_s \in \mathring{S}^{p,1}(\Omega)$ satisfy

$$B(u_s,v) = B(u,v) = F(v), \quad \forall v \in \mathring{S}^{p,1}(\Omega)$$

and we get <u>Theorem 3.3.1</u> [8]. Let Ω^m_σ be the geometrical mesh with the mesh factor $\sigma \in (0,1)$ and let $\mu > 0$ be a degree factor such that $p_{ij} = p_i = 1 + [\mu(i-1)]$ (linear distribution) or $p_{ij} = p = 1 + [\mu m]$ (uniform degree distribution), and $p = \max p_{ij}$ on ${}^c\Omega^m_\sigma$. If $u \in B^2_\beta(\Omega) \cap H^1_D(\Omega)$, is the solution of the problem, then

$$\|u - u_s\|_E \le Ce^{-bN^{1/3}} \qquad (3.3.1) \qquad \square$$

where N is the number of degrees of freedom (i.e. dimension of $S^{p,1}(\Omega)$) and constant C and b are independent of N. We see an analogous result as in Theorem 2.4.2 but without exact specification of the constants C and b.

Let us illustrate Theorem 3.3.1 by a numerical example. Consider the elasticity problem on a cracked domain shown in Fig. 3.3.1. We will assume that the material is homogeneous and Poisson ratio $\nu = 0.3$. The exact solution is the first stress intensity mode and tractions are prescribed on $\Gamma \in \Omega$. Because of the symmetry, we consider only the upper half of the domain. The sequence of the used meshes A_n with $\sigma = 0.15$ and $n = m + 1$ is shown in Fig. 3.3.2. The uniform degree distribution was used with $p = 1 + [\mu m]$, $\mu = 1$. The computation has been made by the program MSC/PROBE. In Table 3.3.1 we show the basic results and the relative error $\|e\|_{E,R} = \frac{\|e\|_E}{\|u_0\|_E} \cdot 100\%$ with b and C in (3.3.1). By $\|u_0\|_E$ we denoted the energy norm of the exact solution. In Fig. 3.3.3 we show the error as function of N. We present here also the error for the p-version. The error of the h-p version $(\delta = 1)$ is shown by the solid line. We see that the line is straight as expected from the theory. The values of constants b and C cannot be at

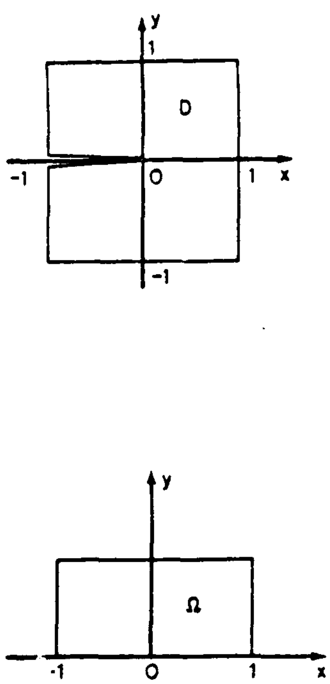

Fig. 3.3.1. Cracked panel.

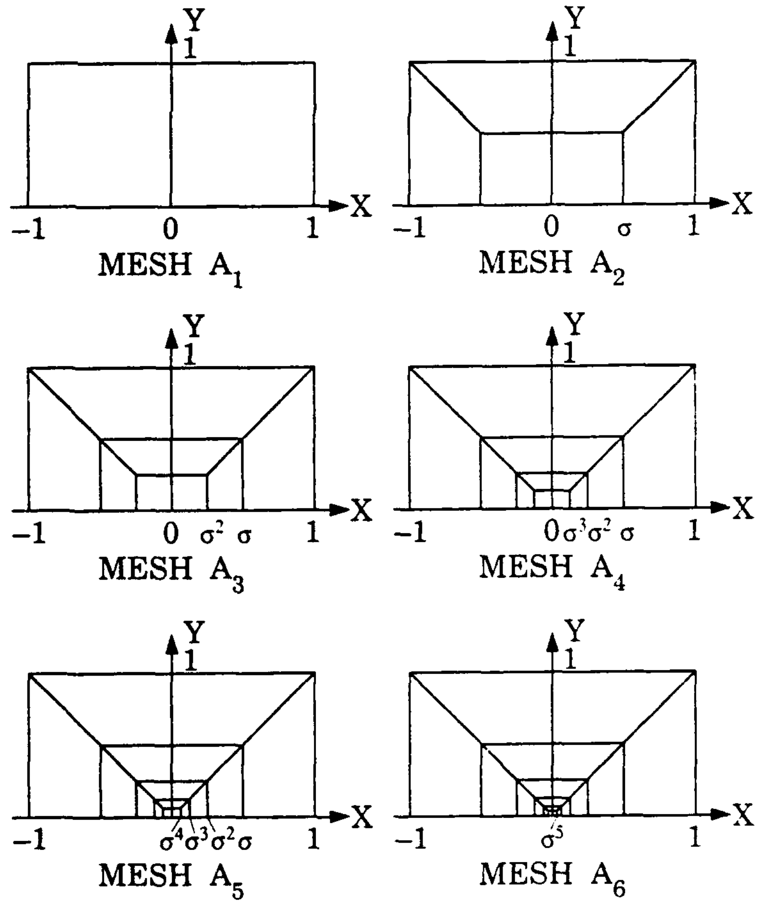

Figure 3.3.2. Geometric mesh A_n. $1 \le n \le b$.

present theoretically determined. In Fig. 3.3.4 we show the error of the h-p version for $\sigma = 1$ and various σ. We see that the value $\sigma = 0.15$ is nearly optimal.

TABLE 3.3.1

Performance of the h-p version on mesh A_n, $1 \le n \le 6$, $\sigma = 0.15$, $\mu = 1$.

Mesh	p	N	$N^{1/3}$	$\Vert e\Vert_{E,R}\%$	b	$C\Vert u_0\Vert_E$
A_1	1	9	2.08	60.92	0.741	1.455
A_2	2	48	3.63	20.23	0.740	2.303
A_3	3	121	4.95	7.61	0.776	2.098
A_4	4	256	6.35	2.57	0.720	1.810
A_5	5	477	7.82	0.90	0.670	1.683
A_6	6	808	9.31	0.33	0.670	1.688

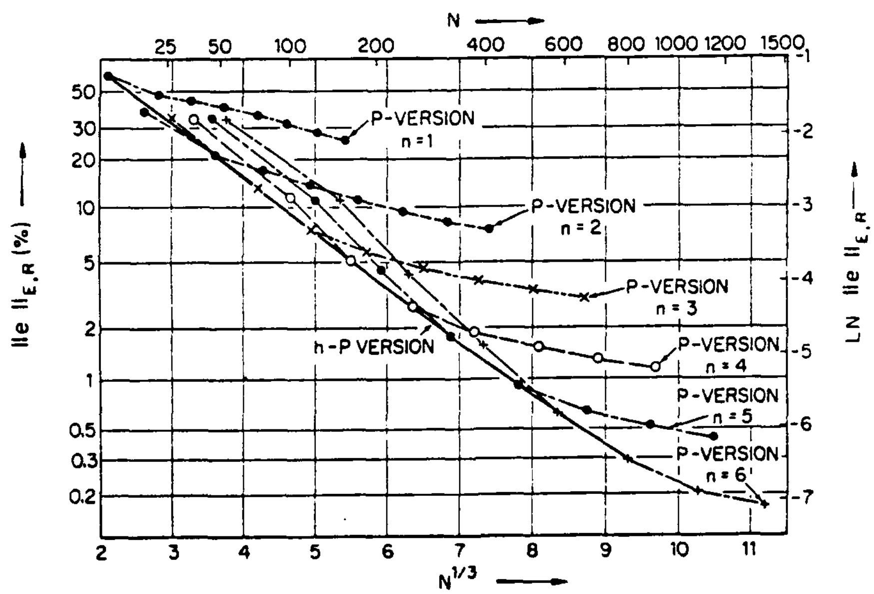

Fig. 3.3.3. Performance of the p and h-p version on Mesh A_n, $1 \le n \le 6$, $\sigma = 0.15$.

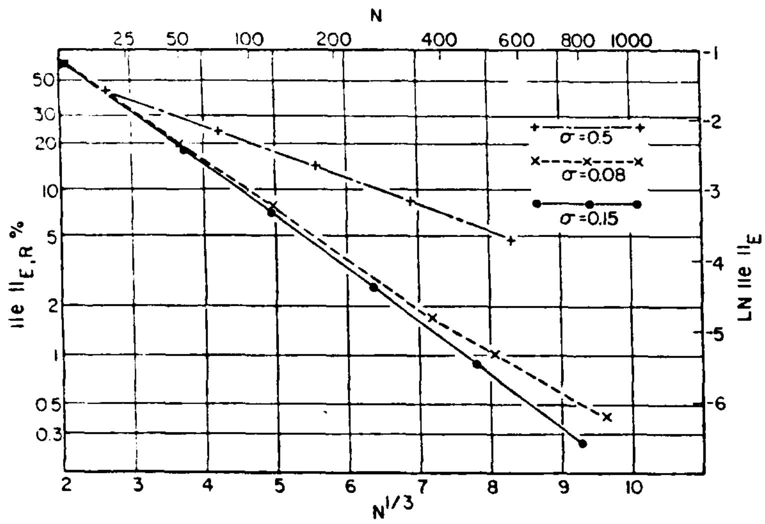

Fig. 3.3.4. Performance of geometric Mesh with σ = 0.15, 0.08 and 0.5.

We note that the results are analogous to those in Section 2. If the degrees of the element are linearly distributed we would get $b' \approx \sqrt[3]{3}\, b$ in (3.3.1) and Table 3.3.1.

The adaptive procedure consists of the design of the estimator E and the adaptive principles. The error estimator can be based on the extrapolation technique. In Table 3.3.2 we show the exact error $\|e\|_E$, the computed error estimator $\mathcal{E} = \|\tilde{e}\|_E$ and the relative error of the estimator in %. We see very high reliability of the estimator. The adaptive approach for the h-p version is a complex one. Here once more the local and global adaptive approach, as explained in Section 2, can be used. In contrast to the one dimensional case transitional elements between the element of different degrees are needed. In addition, various shape functions can be adaptively chosen so that elements are not of a full degree. Further, different type of elements, triangular or quadrilateral can be used and the degree of element can also be understood as a total or tensor product type.

TABLE 3.3.2

Error estimator of the h-p version on Mesh A_n, $1 \le n \le 6$, $\sigma = 0.15$, $\mu = 1$.

$\|\tilde{e}\|_E$	$\|e\|_E$	$\|\tilde{e}\|_{E,R}\%$	$\|e\|_{E,r}\%$	$(\|e\|_E - \|\tilde{e}\|_E)/\|e\|_E\%$
2.9596E-1	2.9662E-1	60.83	60.92	0.2189
9.8774E-2	9.8511E-2	20.28	20.23	-0.2669
3,7033E-2	3.7055E-2	7.606	7.611	0.0606
1.2489E-2	1.2500E-2	2.565	2.567	0.0926
4.3359E-3	4.3691E-3	0.891	0.897	0.6689

The implementation influences very heavily the adaptive process. For some results, we refer here to [2][3]. It seems once more that the h-p adaptive approach based on the adaptively constructed meshes for the h-version and various p (analog to the case addressed in Theorem 2.4.4) is practically not usable. Once more we see that the theoretical results could give a very good insight in the problem of the design of an adaptive procedure. The first theoretical analysis of the h-p version in conjunction with the regularity of the solution described by countably normed spaces appeared in [38]. For more results, we refer to [8][9][10][11][12] and the survey [20] where additional references are available.

3.4. The p version

In the p-version the mesh is fixed and the degrees of the elements are increased uniformly or nonuniformely. If the domain Ω has corners and the solution is singular in the areas of these corners, the rate of convergence of the method is algebraic (see [13] [20] [21] [22]). This is quite analogous behavior results as in one dimension. If $u \in B^2_\beta(\Omega)$ then $\|e\|_E \le CN^{-2(1-\beta)}$. The rate is twice as large as in the case of the h-version with a quasiuniform mesh. In the case that the solution is analytic on $\bar{\Omega}$, we have $\|e\|_E \le$

$e^{-bN^{1/2}} \approx Ce^{-\bar{b}p}$. As in one dimension, the accuracy is for large p governed by the error on the element containing the vertex of the domain, but for smaller p and the refined mesh, the error could be governed by the elements far from the singularity.

As was said in Section 2, the user designs the mesh which is refined in the neighborhood of the corners and other critical places and the degree of the elements is determined adaptively. The mesh which typically has not too many elements is designed by the experience and simple rules. Another way is to use a crude mesh constructed directly by the user or adaptively by the h-method with small p, say 3. Then this mesh could be further refined by the users experience (see e.g. [59]) or by an expert system (see, e.g. [15]). We underline that the degrees of the elements are adaptively determined so that reliable results are obtained for any choice of the meshes; of course, a "bad" mesh will lead to a higher computer cost.

We will assume that the constructed meshes are of geometric type in the neighborhood of the vertices with m_0 layers, i.e., we will consider the mesh $\Omega_\sigma^{m_0}$ with m_0 fixed. From the analysis of [8] we have

$$\|e\|_E \le C\left[\sigma^{2(m_0+2)(1-\beta)} p^{-2(1-\beta)} + Q^p p^\gamma\right] \tag{3.4.1}$$

where $0 < Q < 1$ and $\gamma < 0$.

The first term characterizes the error in the elements closest to the singularities and the second term all others. For large p (as in one dimension), the first term dominates. For small p the second one dominates and the rate is exponential. Hence, the p-version has two phases, asymptotic and pre-asymptotic. The preasymptotic phase is until the elements closest to the singularity is not governing the accuracy. Hence, the user tends to design such a mesh for which the desired accuracy in both terms in (3.4.1) is balanced.

Let us present the computational results for the same elasticity problem as in Section 3.3.3. In Fig. 3.3.3 we already have shown the results of the p-version. In Fig. 3.4.1 we show the relative error in the $\ell n \times \ell n$ scale.

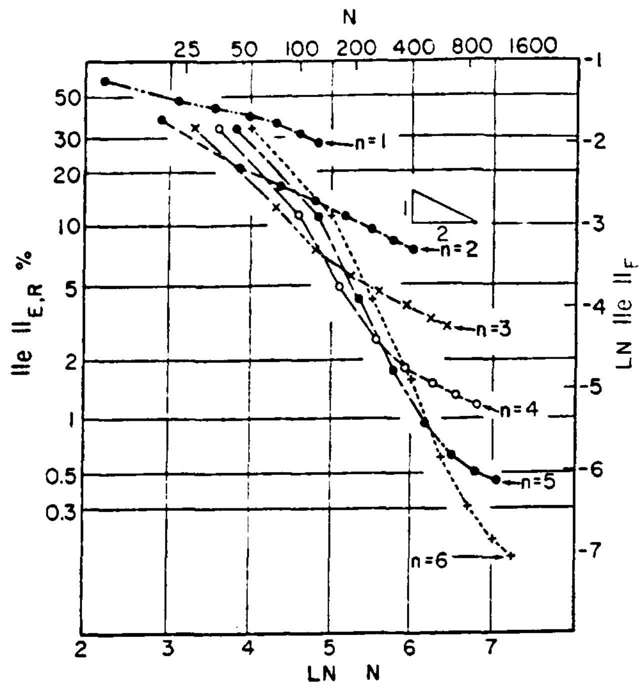

Fig. 3.4.1. The error of the p-version.

We see that for a fixed mesh the accuracy curve has a S shape with the pre-asymptotic and asymptotic phase and inflection pint roughly when the error of the elements closest to the vertex become to be dominant.

Let us now consider the problem for the Laplace equation on the domain shown in Fig. 3.3.1 and the exact solution $u = r^{1/2}\cos \theta/2$, where (r,θ) are the polar coordinates. Because of the symmetry only one half of the domain will be considred. The boundary condition is of Neumann type. On the side $\{-1 < x < 0, \ y = 0\}$ zero Dirichlet condition is prescribed. The used mesh is shown in Fig. 3.3.2 except the two square elements containing singularity are replaced by four equal triangular elements. We use $\sigma = 0.13$ and $\sigma = 0.50$.

In Figs. 3.4.2 and 3.4.3 we show the relative error $\|e\|_{E,R}$ % as function of N in the $\ell n \times \ell n$ scale for different p and various number of layers. The layers are depicted by marks. In Table 3.4.1 and 3.4.2 we show the data ploted in Fig. 3.4.2 and3.4.3. In these tables we indicte by solid lines the numbers of layers when the mesh is overrefined, i.e., when the

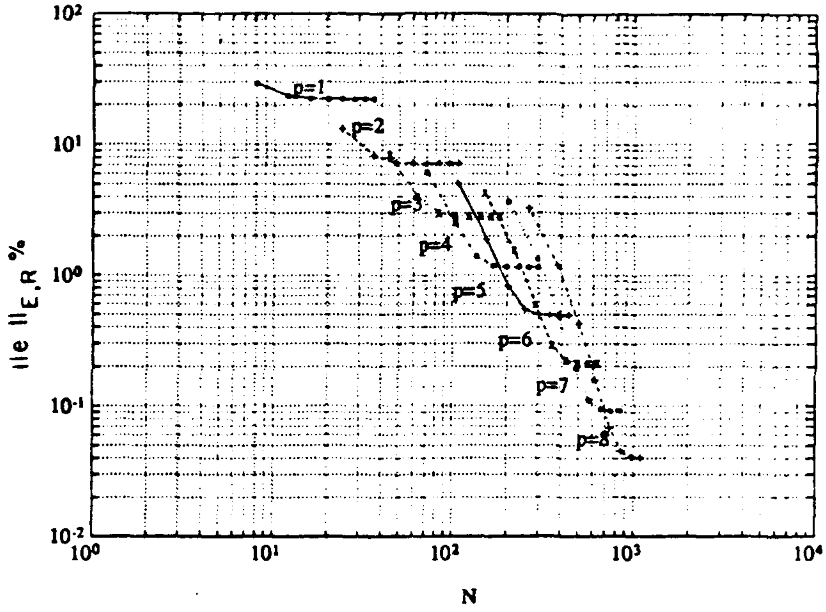

Fig. 3.4.2. The error of the p version for σ = 0.13.

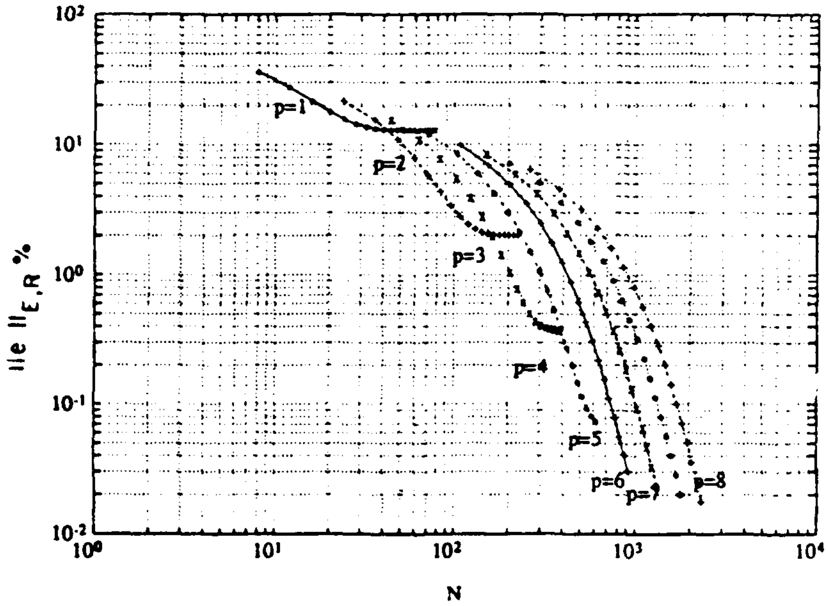

Fig. 3.4.3. The error of the p-version for σ = 0.5

error is not decreasing for the increased number of layers. This, as in one dimension, depends on the degree p of the elements. We see that the number of 3-4 layers and $\sigma = 0.13$ is optimal and gives very good performance. For p increasing the case $\sigma = 0.5$ is also leading to the convergent results but less efficiently. So we see that overrefinement is desirable.

In Tables 3.4.3, 3.4.4 and 3.4.5 we compare the performance of the meshes for different ranges of accuracy.

Tables 3.4.3 - 3.4.5 show well the influence of the number of layers on the accuracy. We emphasize that the element degree selection leading to the desired accuracy is made adaptively. In practice $m \approx 3$ is usually used and here $\sigma \approx 0.15$ is a good choice for the accuracies of 1-3%. To show it, we present in Table 3.4.6 the achieved accuracy for $m = 3$ as function of p and N for $\sigma = 0.13$ and $\sigma = 0.50$.

The computations results we have presented are in very good agreement with the theoretical analysis.

TABLE 3.4.1. Performance of the p version on geometric mesh with σ = 0.13

Mesh A_n	Number of layer, m	p = 1		p = 2		p = 3		p = 4	
		N	$\lVert e\rVert_{E,R}$%	N	$\lVert e\rVert_{E,R}$%	N	$\lVert e\rVert_{E,R}$%	N	$\lVert e\rVert_{E,R}$%
A_2	1	8	29.13	24	13.03	44	8.33	72	6.21
A_3	2	12	23.53	36	8.20	64	4.00	104	2.49
A_4	3	16	22.48	48	7.28	84	2.98	136	1.40
A_5	4	20	22.29	60	7.15	104	2.82	168	1.19
A_6	5	24	22.72	72	7.13	124	2.80	200	1.15
A_7	6	28	22.26	84	7.13	144	2.80	232	1.15
A_8	7	32	22.26	96	7.13	164	2.80	264	1.15
A_9	8	36	22.26	108	7.13	184	2.80	296	1.15

Mesh A_n	Number of layer, m	p = 5		p = 6		p = 7		p = 8	
		N	$\lVert e\rVert_{E,R}$%	N	$\lVert e\rVert_{E,R}$%	N	$\lVert e\rVert_{E,R}$%	N	$\lVert e\rVert_{E,R}$%
A_2	1	108	5.03	152	4.24	204	3.67	264	3.24
A_3	2	156	1.87	220	1.54	296	1.33	384	1.17
A_4	3	204	0.81	288	0.59	388	0.48	504	0.42
A_5	4	252	0.54	356	0.29	480	0.19	624	0.15
A_6	5	300	0.547	424	0.22	572	0.11	744	0.068
A_7	6	348	0.49	492	0.21	664	0.094	864	0.046
A_8	7	396	0.49	560	0.21	756	0.093	984	0.041
A_9	8	444	0.49	268	0.21	847	0.092	1104	0.040

TABLE 3.4.2. Performance of the p version on geometric mesh with σ = 0.50

Mesh A_n	Number of layer, m	p = 1		p = 2		p = 3		p = 4	
		N	$\|e\|_{E,R}\%$	N	$\|e\|_{E,R}\%$	N	$\|e\|_{E,R}\%$	N	$\|e\|_{E,R}\%$
A_2	1	8	36.01	24	21.40	44	15.34	72	11.96
A_3	2	12	27.45	36	15.30	64	10.89	104	8.48
A_4	3	16	21.60	48	10.95	84	7.72	136	6.00
A_5	4	20	17.85	60	7.88	104	5.47	168	4.24
A_6	5	24	15.58	72	7.76	124	3.88	200	3.00
A_7	6	28	14.30	84	4.31	144	2.75	232	2.12
A_8	7	32	13.60	96	3.36	164	1.96	264	1.50
A_9	8	36	13.23	108	2.76	184	1.41	296	1.06
A_{16}	9	40	12.95	120	2.41	204	1.03	328	0.75
A_{11}	10	44	12.95	132	2.21	224	0.77	360	0.53
A_{12}	11	48	12.90	144	2.11	244	0.60	392	0.38
A_{13}	12	56	12.86	168	2.02	284	0.44	456	0.20

TABLE 3.4.2. Performance of the p version on geometric mesh with $\sigma = 0.50$.
(Continued)

Mesh A_n	Number of layer, m	p = 5		p = 6		p = 7		p = 8	
		N	$\Vert e\Vert_{E,R}\%$	N	$\Vert e\Vert_{E,R}\%$	N	$\Vert e\Vert_{E,R}\%$	N	$\Vert e\Vert_{E,R}\%$
A_2	1	108	9.80	152	8.30	204	7.20	264	6.35
A_3	2	156	6.94	220	5.87	296	5.09	384	4.49
A_4	3	204	4.91	288	4.16	388	3.60	504	3.18
A_5	4	252	3.47	356	2.29	480	2.55	624	2.25
A_6	5	300	2.45	424	2.08	572	1.80	744	1.59
A_7	6	348	1.74	492	1.47	664	1.27	864	1.126
A_8	7	396	1,23	560	1.04	756	0.90	984	0.79
A_9	8	444	0.87	628	0.73	848	0.64	1104	0.56
A_{16}	9	492	0.61	696	0.52	940	0.45	1224	0.40
A_{11}	16	540	0.43	764	0.37	1032	0.32	1344	0.28
A_{12}	11	588	0.31	832	0.26	1124	0.23	1464	0.20
A_{13}	12	588	0.15	968	0.13	1308	0.11	1704	0.10

TABLE 3.4.3. Comparison of performance of the p-version on Ω_σ^m for the accuracy 1%.

p	$\sigma = 0.13$			$\sigma = 0.50$		
	m	N	$\Vert e\Vert_{E,R}\%$	m	N	$\Vert e\Vert E,R\%$
3				9	204	1.03
4	4	168	1.19	8	296	1.06
5	3	204	0.81	8	444	0.87
6	2	220	1.54	7	560	1.04
7	2	296	1.33	7	756	0.90
8	2	384	1.17	6	864	1.12

TABLE 3.4.4. Comparison of the p-version performance on Ω_{σ}^{m} for the accuracy 3%

p	$\sigma = 0.13$			$\sigma = 0.5$		
	m	N	$\|e\|_{E,R}\%$	m	N	$\|e\|_{E,R}\%$
2				8	108	2.76
3	3	84	2.98	6	144	2.75
4	2	104	2825	5	200	3.00
5	2	156	1.87	5	300	2.45
6	2	220	1.54	4	356	2.94
7	2	296	1.33	4	480	2.55
8	1	264	3.24	3	504	3.18

TABLE 3.4.5. Comparison of the p-version performance on Ω_{σ}^{m} for the accuracy 12% .

p	$\sigma = 0.13$			$\sigma = 0.15$		
	m	N	$\|e\|_{E,R}\%$	m	N	$\|e\|_{E,R}\%$
1				10	44	12.95
2	1	24	13.03	3	48	10.95
3	1	44	8.33	2	64	11.89
4	1	72	6.21	1	72	11.96
5	1	108	5.03	1	108	9.80
6	1	152	4.24	1	152	8.30
7	1	204	3.67	1	204	7.20
8	1	264	3.24	1	264	6.35

TABLE 3.4.6. Comparison of performance of the p version on Ω_σ^3.

p	N	$\sigma = 0.13$	$\sigma = 0.50$
		$\|e\|_{E,R}\%$	$\|e\|_{E,R}\%$
1	30	22.47	21.10
2	60	7.28	10.95
3	104	2.97	7.72
4	168	1.37	6.00
5	252	0.76	4.91
6	356	0.51	4.16
7	480	0.38	3.60
8	674	0.30	3.18

TABLE 3.4.7. Performance of Error Estimator on Mesh Ω_σ^m, $\sigma = 0.13$, $m = 2$.

p	N	E_R %	$\|e\|_{E,R}\%$	θ
1	12	23.52	23.56	1.00
2	36	8.19	8.26	0.999
3	64	3.97	4.00	0.992
4	104	2.45	2.49	0.984
5	156	1.82	1.87	0.973
6	220	1.47	1.34	0.954
7	296	1.25	1.32	0.947
8	384	1.08	1.17	0.923

As we said, the essential part of the code is an error estimator. For the p-version the estimator based on the extrapolation is very effective. The quality of the estimator can be judged by its effectivity index θ

$$\theta = \frac{\text{estimate}}{\text{true error}} = \frac{\mathcal{E}}{\|e\|}$$

which can be computed by in benchmark computations. We show in Table 3.4.7 the quality of the estmator for on example when the mesh A_3 (i.e., Ω^m_σ, $m = 2$) is used for $\sigma = 0.13$.

3.5. The adaptive p and h-p versions

The aim of the adaptive approach is to obtain (in a most effective way) the solution in the a-priori given range of accuracy with a reliable error estimate. The effectiveness is meant in general way, i.e., includes the computer and manpower cost. The adaptive approach is very much influenced by the implementation aspects. The code based on the h, p and h-p version have very different characters. Here we will concentrate on the p version and the error measure of the energy norm.

1) The mesh generation is for complex geometry very laborious. Various mesh generators commercial and not commercial are available today. They focus primarily on the handling of the geometry. In the case of the h-version (in two dimensions) various adaptive codes are available. Nevertheless, these meshes are not appropriate for the p-version (although, of course, they can be used too).

2) The mesh for the p-version is characterized by large elements with a (geometric) refinement in the neighborhood of singularities.

3) The mesh in the neighborhood of the singularity has to be rather overrefined than underrefined. The underrefinement can be more "risky". The reasonable overrefinement is not costly. The underrefinement can lead to pollution effects. For the stronger singularity more layers have to be used.

Usually we will recommend 2-4 layers., The factor σ of the geometrical mesh in the neighborhood of the corners can be different, nevertheless, we recommend, for the reasons explained in previous sections, to use $\sigma \approx 0.15$.

4) The mesh generator has to have special features to lead to the meshes which are advantageous for the p-version. They can be based on various principles. For example, on the design of the crude mesh and special "layer" refinement in the neighborhood of the critical places possibly by help of an expert system (see, e.g. [15]), etc.

5) If the information about the solution in some point the neighborhood of the corner is needed, then these points have to be separated from the corner by 2 layers. The stresses in the area very close to the corners have to be computed by stress intensity factors.

6. The p-version could suffer by an oscillation of the solution if the mesh is improperly designed. This could happen in the elements which contain the singularity. This element has to be small and direct stress computation in this elements should not be used, We note that the energy error converges also in this element. Nevertheless, experience backed by the theory shows that for proper mesh no oscillation occurs.

7) The error indicators and estimators based on the comparison of data for various consecutive p performs very well, not only for the energy norm accuracy measure. This is essential because the energy norm measure is usually not the most important one and possibly could be misleading when other accuracy measure is of interest.

8) The error indicator can be used elementwise. This is essential if the accuracy will not be achieved in the range of admissible range of degrees p. Then a new mesh is created by refinement of the elements with largest error indicators.

9) So far we mentioned only the p adaptivity with uniform p. The

adaptive approach can be based on adaptive additions of various shape functions so that the notion of degrees is loosing its meaning.

10) The experience holds also for the p-version that quadrilateral elements are better than the triangular ones.

11) One of the main advantage of the p-version is relatively easy implementation and possibility of the reliable error control for any data of interest by comparing the computed data for various p.

12) The implementation and adaptive strategy for the h-p version is much more complicated but could be made, see, e.g. PHLEX program [49] [50]. The possible plausible approach to construct the h-p version by optimal meshes for the h-version for increasing p cannot be recommended for practical reasons.

13) For additional aspects of the p-version for the analysis of complicated engineering problems by the STRIPE program, we refer to [2].

4. CONCLUSIONS

We presented basic ideas behind the h, p and h-p versions and characteristic theoretical results. The comparison of these approaches is a complex task dependent on the criteria used. It reflects the implementation aspects, computer hardware, etc. By our opinion one of the basic comparison criterion has to be the possibility of accurate and reliable quantitative assessment of the accuracy of <u>any data</u> of engineering interest. It seems that the adaptive p-version (on a properly designed mesh) is one of the best approach for the practical engineering computations of problems in structural mechanics. (See also [2].) Of course, changing criteria of comparison or a class of the problem could lead to different conclusions.

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The Laboratory for Numerical Analysis is an integral part of the Institute for Physical Science and Technology of the University of Maryland, under the general administration of the Director, Institute for Physical Science and Technology. It has the following goals:

- To conduct research in the mathematical theory and computational implementation of numerical analysis and related topics, with emphasis on the numerical treatment of linear and nonlinear differential equations and problems in linear and nonlinear algebra.

- To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.

- To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.

- To assist with the education of numerical analysts, especially at the postdoctoral level, in conjunction with the Interdisciplinary Applied Mathematics Program and the programs of the Mathematics and Computer Science Departments. This includes active collaboration with government agencies such as the National Institute of Standards and Technology.

- To be an international center of study and research for foreign students in numerical mathematics who are supported by foreign governments or exchange agencies (Fulbright, etc.).

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